Math 53, Discussions 116 and 118 **2022-8-31**

More calculus with parametric curves

Answers included

Questions

Question 1. Recall the formulas for area and arclength for parametric curves. Make sure you understand how to use them—for example, are there any assumptions you should make regarding how the curve is traced out?

Question 2. Sketch the graph $y = \ln x$ between the points $(1, 0)$ and $(e, 1)$. Rewrite this as a parametric curve (remember that every graph has a "canonical" parametric representation).

(a) Write down an integral which computes the area underneath this curve and above the x-axis.
(b) Write down an integral which computes the area to the left of this curve and to the right of the y-axis.
(c) Observe that the two areas together form a rectangle. Use this to relate the two integrals from the preceding parts, and then compute the value of the integral from part (a) via your equation. **What is the name of this integration technique?**

Question 3 (Stewart §10.2 #51)**.** Find the distance traveled by a particle with position (x, y) as t varies in the given time interval. Compare with the length of the curve.

$$x = \sin^2 t \qquad y = \cos^2 t \qquad 0 \le t \le 3\pi$$

(The purpose of this question is to remind you that these two quantities can be different. You can actually do this particular problem without any calculus.)

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 2. We can parametrize the curve as $x = t$, $y = \ln t$ for $1 \le t \le e$.

(a) The integral is

$$\int_1^e (\ln t) \cdot 1 \, dt.$$

(b) The integral is

$$\int_1^e t \cdot \frac{1}{t} \, dt.$$

(c) The equation yields

$$\int_1^e \ln t \, dt = e - \int_1^e dt = 1.$$

This is none other than **integration by parts**. I explained in discussion why this is the case.

Question 3. The curve is just the line segment from $(1, 0)$ to $(0, 1)$. You can see this by noting the simple Cartesian equation $x + y = 1$. In the period $0 \le t \le 3\pi$, the particle traverses the segment 6 times. So the total distance traveled is $6\sqrt{2}$, whereas the length of the curve is just $\sqrt{2}$.

That's how you can see the answer without calculus. To do this with calculus, the integral which computes the distance traveled is

$$\int_0^{3\pi} \sqrt{2\sin(2t)^2} \, dt.$$

However I demonstrated in class that you need to be careful when simplifying this:

$$\int_0^{3\pi} \sqrt{2} \, |\sin(2t)| \, dt$$

(note the absolute values). Then by a periodicity argument the above integral is equal to

$$6 \int_0^{\pi/2} \sqrt{2} \sin(2t) \, dt = 6\sqrt{2}.$$